Tchebychev filter with a bandwidth of 20 Mc/s centered upon 6.900 Gc/s. The ground plane spacing b was chosen to be 0.5 inch, which resulted in a finger diameter d of 0.199 inch, and the spacing x was chosen to be 0.125 inch. From Fig. 2, the fringing capacitance (C_f) was found to be 0.132 pF, and from (2) the "parallel plate" capacitance (C_p) was 0.055 pF, giving a total end capacitance (C_t) of 0.187 pF.

By rearranging (1),

$$l = \frac{c}{\omega_0} \tan^{-1}\left(\frac{1}{\omega_0 Z_0 C_t}\right) \tag{3}$$

and, hence, the finger length could be calculated. The measured center frequency of this filter was found to be 6 896 Gc/s, whereas scaling from the design adopted by Matthaei[1] and Cristal[2] resulted in a filter having a center frequency of 7.00 Gc/s. A number of existing filters in the frequency range 500 Mc/s to 7 Gc/s have been analyzed by graphically solving (1), which is a transcendental equation in ω, for each mechanical structure. The center frequencies calculated by this method would have reduced the discrepancies between design and measurement by at least 70 percent in all cases. As an example, the filter quoted by Cristal[2] had a design center frequency of 1.500 Gc/s. By applying the above procedure, a center frequency of 1.543 Gc/s would be predicted, which is in close agreement with the measured center frequency 1.557 Gc/s.

It is interesting to note that to obtain a particular resonant frequency for a given end plate separation, the length of each finger in a filter is a function of the diameter. Since the fingers in the center of these filters are often of equal diameter, they will all have the same resonant frequency. However, the fingers at the ends of the filter are usually of different diameters, and ideally the finger lengths should be adjusted accordingly. This adjustment is not usually made and provides an alternative explanation of the slight adjustments in coupling (and effectively Z_0) which always seem to be necessary between the end elements.

In conclusion, it is felt that, whereas a more rigorous analysis of these discontinuities would undoubtedly lead to even closer agreement, the above procedure should prove useful in reducing considerably the gap between the design and measured center frequencies of interdigital filters.

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Application of the Point-Matching Method in Waveguide Problems

The determination of cutoff frequencies of waveguides with arbitrary cross section by the point-matching technique was recently studied by Yee and Audeh [1], [2].

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It is interesting to point out that the point-matching technique or collocation method is a well-known and popular procedure in several areas of applied mechanics. It has been used in boundary value and eigenvalue problems [3]–[11]. The problem treated by Baltrukonis [9] is governed by the same differential system which governs the propagation of electromagnetic waves in hollow-piped waveguides. Baltrukonis [9] deals with a star-shaped boundary given by the equation

$$S(r, \theta) \equiv r - (a + b \cos 4\theta) = 0.$$

The circle is one curve of the family. The first four eigenvalues were calculated and plotted in function of the dimensionless parameter b/a. When b/a approaches zero the boundary is circular, for which the exact solution is known. This study shows that, for the problem under consideration, the calculated eigenvalues depend rather drastically on the distribution of points. Furthermore, little or no convergence is demonstrated for as many as seven collocation points taken within an octant of the boundary.

Jain [10] has introduced a new criteria for the collocation procedure. In this procedure one requires that the error at adjacent matching points be equal in magnitude but opposite in sign. Furthermore, the error at the matching points must be larger than that at any other point. This technique seems to yield better results than the straight collocation method [11].

One of the main advantages of the point matching is its simplicity. On the other hand, it should be emphasized that many uncertainties exist regarding accuracy of the results when the method is applied to new problems.

Indication of convergence can usually be obtained by using conformal mapping along with various approximation techniques [12]–[14]. One of the main advantages of using conformal mapping is that some bounding techniques can generally be used once the boundary conditions are identically satisfied [13]–[15]. It was shown [14] that the use of conformal mapping and Galerkin's Method leads to excellent results.

In summary, it is the opinion of the author that additional study of all these approximate techniques from a unified point of view is needed.

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Resonances of a Cylindrical Cavity in a Lossy Compressible Plasma

Wait [1] has derived equations describing the resonances of a cylindrical cavity in an isotropic, lossy compressible plasma. The purpose of this communication is to consider low-frequency resonances when the effects of losses and compressibility are small.

Consider a cylindrical cavity which is a free space region immersed in an isotropic compressible lossy plasma. The plasma has permeability equal to the free space value μ_0 and permittivity ϵ where

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{\omega_N^2}{(\nu + i\omega)i\omega}. \tag{1}$$

In (1), ϵ_0 is the permittivity of free space, ω_N is the angular plasma frequency of the electrons, and ν is the electron collision frequency. The fields have the time factor $e^{i\omega t}$, where ω is the angular frequency and t is the time. Let [1]

$$\tau_p^2 = -\frac{\omega^2}{u^2}\left(1 + \frac{\nu}{i\omega}\right)\frac{\epsilon}{\epsilon_0} \tag{2}$$

where u is the speed of electron acoustic waves in the plasma.

Suppose that the dimensions of the cavity are small compared with the electromagnetic wavelength both in free space and in the plasma. That is, $|\tau a| \ll 1$ and $|\tau_e a| \ll 1$, where $\tau^2 = -\omega^2\mu_0\epsilon_0$, $\tau_e^2 = -\omega^2\mu_0\epsilon$, and a is the radius of the cavity. Then the resonance condition is [1]

$$\frac{\epsilon}{\epsilon_0} + (1 - \delta_n) = 0 \tag{3}$$

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where δ_n is defined by equation (65) of Wait [1]. It is easily found that for $|\tau_e a| \ll 1$:

$$\delta_n = -\frac{in\omega_N^2}{\omega(\nu + i\omega)} \cdot \frac{1}{\tau_p a} \cdot \frac{K_n(\tau_p a)}{K_n'(\tau_p a)} \cdot \quad (4)$$

In (4), n is an integer, K_n is a modified Bessel function, and a dash denotes differentiation with respect to the argument. Equations (3) and (4) describe the low-frequency resonances of the cylindrical cavity. They may be compared with corresponding expressions which have been derived for a spherical cavity [2].

For an incompressible plasma $u=0$. Then τ_p is infinite and $\delta_n=0$. In this case, the resonance condition (3) becomes $\epsilon/\epsilon_0=-1$ [1], [3]. Budden [3] has considered the low-frequency resonances of a cylindrical cavity in an incompressible anisotropic loss-free plasma. The effect of losses in this situation has also been considered [4].

For $|z| \gg 1$ and $|z| \gg |\nu|$, the modified Bessel function $K_\nu(z)$ of complex order ν and complex argument z satisfies

$$K_\nu(z) \sim (\pi/2z)^{1/2} e^{-z}, \quad -\pi < \arg z < \pi/2 \quad (5)$$

[5], so that $K_\nu'(z)/K_\nu(z) \sim -1$. Hence, for $|\tau_p a| \gg 1$ and $|\tau_p a| \gg |n|$, using (2) and (1), (4) becomes

$$\delta_n \doteqdot \frac{n\omega_N^2}{\omega^2(\nu + i\omega)} \frac{u}{a} \left[1 - \frac{\omega_N^2}{\omega^2} - \frac{i\nu}{\omega}\right]^{-1/2} \cdot \quad (6)$$

This is applicable when the dimensions of the cavity are much greater than the wavelength of the electron acoustic waves in the plasma.

A low-frequency resonance when the plasma is incompressible and loss-free will satisfy (3) with $\delta_n=0$ and $\nu=0$. The only such frequency is $\omega_N/2^{1/2}$ [3]. Thus, for a given electron density, there is only a single resonant frequency, and this frequency is independent of the radius of the cavity. This result may be compared to that for a spherical cavity in an incompressible and loss-free plasma, for which there is an infinite number of resonances [2], [3].

When ν/ω is small and $|\tau_p a|$ is large, the effects of losses and compressibility will be small. The resonant frequencies can then be regarded as having been slightly perturbed from that in the incompressible loss-free case.

Let $(\omega_N/2^{1/2})+\Omega$, where $|\Omega| \ll \omega_N/2^{1/2}$, be a (complex) resonant frequency when the effects of losses and compressibility are small. When $\nu/\omega \ll 1$, (1) becomes

$$\frac{\epsilon}{\epsilon_0} \doteqdot 1 - \frac{\omega_N^2}{\omega^2} - \frac{i\nu\omega_N^2}{\omega^3} \cdot \quad (7)$$

Neglecting the product $w\nu$ in (6) gives

$$\delta_n \doteqdot \frac{n\omega_N^2}{i\omega^3} \frac{u}{a} \left(1 - \frac{\omega_N^2}{\omega^2}\right)^{-1/2} . \quad (8)$$

Hence, the resonance condition (3) can be written

$$2 - \frac{\omega_N^2}{\omega^2} \doteqdot \frac{i\nu\omega_N^2}{\omega^3} + \delta_n \quad (9)$$

where δ_n is given by (8). Replacing ω in the right-hand side of this expression by its unperturbed resonant value $\omega_N/2^{1/2}$, and using the condition $|\Omega| \ll \omega_N/2^{1/2}$ in the left-hand side, gives

$$\Omega = \frac{i\nu}{2} - \frac{nu}{2a} \cdot \quad (10)$$

Hence, in a shock-excited resonance of the cavity the fields vary with time as

$$\exp\left[i\left(\frac{\omega_N}{2^{1/2}} - \frac{nu}{2a}\right)t\right] \cdot \exp\left(-\frac{\nu}{2}t\right). \quad (11)$$

Thus, the time constant with which the fields decay depends only on the electron collision frequency; it is independent of the compressibility and the mode number n. This time constant is the same as that found for a spherical cavity in an isotropic slightly lossy, slightly compressible plasma [2].

The fields oscillate with a real frequency which is independent of the losses. The effect of a non-zero value of u is to split the unperturbed resonant frequency $\omega_N/2^{1/2}$ into a series of resonant frequencies, each separated from the next by the amount $u/(2a)$. Of course, for sufficiently large $|n|$, the quantity $|n|u/(2a)$ will no longer be small compared with $\omega_N/2^{1/2}$; then (11) will not be applicable.

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Attenuation Constants of Waveguides with General Cross Sections

I. Theory

The cutoff frequencies and the fileld configurations of waveguides with general cross section can be calculated approximately by the point-matching method,[1] provided that the method is applicable. With the field configurations of the ideal waveguide (with perfectly conducting guide walls) known, it is expected that the attenuation constant due to the finite conductivity of the guide walls may be estimated numerically.

Conventionally, the attenuation constant is defined as

$$\alpha = P_L/2P_T \quad (1)$$

if the guide is made of good conducting material, where P_L is the power loss per unit length. The power transfer P_T is given as[2]

$$P_T = (1/2) \int_S \mathrm{Re}\,[\overline{E}_t \times \overline{H}_t^* \cdot \bar{z}]\, dS \quad (2)$$

where S is the cross-sectional area of the waveguide, $\bar{z}$ is the unit vector in the propagating direction, and (*) denotes the operation of taking the complex conjugate. The transverse components of the field E_t and H_t can be calculated from the longitudinal component ψ ($\psi=H_z$ for TE modes, $\psi=E_z$ for TM modes), which was obtained by the point-matching method.[1] Substituting the expressions of E_t and H_t in terms of ψ into (2), and after some manipulation, the power transfer can be reduced to

$$P_T = G \int_S |\psi|^2 dS \quad (3)$$

where

$$G = (1/2Z_0)(f/f_c)^2 \zeta \quad \text{for TM modes}$$
$$G = (Z_0/2)(f/f_c)^2 \zeta \quad \text{for TE modes}$$
$$\zeta = \sqrt{1 - (f_c/f)^2}$$

and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance of free space. The quantities f_c and f are the cutoff and operating frequencies, respectively.

The power loss per unit length of the guide is conventionally estimated by

$$P_L = (R_s/2) \oint_C |H_{\tan}|^2 dl \quad (4)$$

where $R_s = \sqrt{\omega\mu_0/2\sigma}$ is the surface resistance of the guide wall and σ is the conductivity of the conducting material. The path C of the line integral is the contour of the cross section. The integrand in (4) is the square of the magnitude of the magnetic field component tangential to the periphery of the ideal guide walls. *Since the normal component of the transverse magnetic field H_t automatically vanishes at the guide surface*, it is then possible to express $H_{\tan}$ for TM wave modes as follows:

$$|H_{\tan}|^2 = |\overline{H}_t(r_c, \theta)|^2 \quad (5)$$

where r_c, a function of θ, describes the cross-sectional contour. For TE wave modes, however, the longitudinal component of the magnetic field also contributes to the tangential component. Hence,

$$|H_{\tan}|^2 = |\overline{H}_t(r_c, \theta)|^2 + |\psi(r_c, \theta)|^2. \quad (6)$$

The square of the magnitude of the transverse magnetic field may be written as

$$|\overline{H}_t|^2 = (f/f_c)^2 F(r, \theta) \quad \text{for TM modes} \quad (7)$$

and

$$|\overline{H}_t|^2 = (f\zeta/f_c k^2)^2 F(r, \theta) \quad \text{for TE modes} \quad (8)$$

where

$$F(r, \theta) = \left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2.$$

Combining (1) through (8) yields the following attenuation constants:

$$\alpha = \left(R_s/2Z_0\zeta k^2 \int_S |\psi|^2 dS\right) \oint_C F(r_c, \theta) r_c\, d\theta \quad (9)$$

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